



Epidemic SIS model in air-polluted environment

Tran Dinh Tuong¹

Received: 26 January 2020 / Published online: 31 March 2020

© Korean Society for Informatics and Computational Applied Mathematics 2020

Abstract

In this paper, the epidemic model SIS under polluted environment is investigated. Study of epidemic models often focuses on determining under what conditions the disease will disappear or it will persist. With new techniques developed for periodic stochastic differential equations, we are able to classify the longtime behavior of the epidemic model by introducing a threshold value λ . To be more specific, we show that if $\lambda < 0$ then the disease-free is globally asymptotic stable, i.e., the disease will eventually disappear while the epidemic is strongly stochastically permanent provided that $\lambda > 0$. We also give some of numerical examples to illustrate our results. Moreover, the approach in this paper can be used to more general models.

Keywords SIS epidemic models · Extinction · Permanence · Asymptotic stable

Mathematics Subject Classification 37A25 · 60J60 · 60J75

1 Introduction

The commonly used epidemic models nowadays, in which the density functions are spatially homogeneous, were first introduced in 1927 by Kermack and McKendrick in [16], known as compartment models. Compartmental models are a technique used to simplify the mathematical modeling of infectious disease. The population is divided into compartments, with the assumption that every individual in the same compartment has the same characteristics. Then the dynamics of these classes are formulated by use of a system of deterministic differential equations.

One of classic epidemic models is the SIR model, which subdivides a homogeneous host population into three epidemiologically distinct types of individuals, the susceptible, the infective, and the removed, with their population sizes denoted by S , I and R , respectively. Precisely, the model assumes that the susceptible class can transform

✉ Tran Dinh Tuong
trandinhthuong@gmail.com

¹ Faculty of Basic Sciences, Ho Chi Minh University of Transport, 2 Vo Oanh, Ho Chi Minh, Vietnam

into the infective class through the contact with infected persons, and the infectives can be recovered through treatment so that have permanent immunity. It is suitable for some infectious diseases of permanent or long immunity, such as chickenpox, smallpox, measles, etc. However, some research showed that some diseases, such as transmission dynamics of Ebola virus disease [36], viral diarrhea [9] and hand, foot and mouth disease [4], do not confer any long lasting immunity. The infections do not give immunization upon recovery from infection, and individuals become susceptible again. Individuals have repeat or reoccurring infections, and infected individuals return to the susceptible state. This model is known as the SIS model, see [17]. A differential equation model for this is described by following system of equations

$$\begin{aligned} dS(t) &= (\omega - \beta S(t)I(t) - \mu S(t) + \gamma I(t))dt \\ dI(t) &= (\beta S(t)I(t) - (\mu + \gamma - \gamma_0)I(t))dt \end{aligned}$$

where $S(t)$ and $I(t)$ are the densities of susceptible and infected class of individuals, respectively. In the above, the recruitment rate ω , the infection rate β , the natural death rate μ , the death rate due to the disease γ_0 and the treatment cure rate γ are positive constants. Such above models and related problems have been widely studied; for example, see [5–8, 12, 24, 25, 27, 28, 35] for stability analysis; see [30, 31] for the investigations of exact analytical solutions using a Lie algebra approach applied to the Kolmogorov forward equation of a Markov chain modeling the epidemic dynamics.

Recently, the impact of air pollution on people's health and daily activities has risen much attention. As a consequence, much effort is devoted to data analyzing, modeling the air quality index (AQI), see e.g., [3, 10, 13, 34] and reference therein. By using the real data (such as: 6 year long time series of air quality index (AQI) data, gathered from air quality monitoring sites in Xi'an from 15 November 2010 to 14 November 2016) and method in Statistics (for instant, Bayesian approach and the Gibbs sampler algorithm) as well as techniques in mathematical modeling, the dynamics of AQI is formulated formally by stochastic differential equation (SDE). In details, the SDE model for AQI (see e.g., [10]) is as follows

$$dX(t) = [c(t) - d(t)X(t)]dt + \sigma(t)\sqrt{X(t)}dW(t), \quad (1.1)$$

where $c(t)$ denotes the rate of inflow of pollutants into the air, $d(t)$ is the rate of clearance of pollutants, and $\sigma(t)$ is its volatility. Due to the seasonal effects, we assume that $c(t)$, $d(t)$, $\sigma(t)$ are continuous and periodic with period T .

The impact of environment on the people's health makes effects on the dynamics of infection in general and infection rate β in particular; see e.g., [1, 10, 34]. Under pollution, the infected rate is not only time-inhomogeneous, but also is dependent of the AQI. Roughly speaking, with a good AQI, people should be infected rarely and then, the rate β will be low. In contrast, when the AQI is bad, the infection will happen often, i.e., the infected rate should be higher. Therefore, in polluted environment, the dynamics $X(t)$ describing AQI is added to the classical SIS model and the rate function $\beta(X(t))$ is used, instead of using constant β . To be specific, we consider the following model

$$\begin{cases} dX(t) = [c(t) - d(t)X(t)]dt + \sigma(t)\sqrt{X(t)}dW(t) \\ dS(t) = [\omega - \beta(X(t))S(t)I(t) - \mu S(t) + \gamma I(t)]dt \\ dI(t) = [\beta(X(t))S(t)I(t) - (\mu + \gamma + \gamma_0)I(t)]dt. \end{cases} \quad (1.2)$$

Our main goal in this paper is to provide a sufficient and almost necessary condition for strongly stochastically permanent and extinction of the disease in the stochastic SIS model (1.2). Concretely, we establish a threshold λ such that the sign of λ determines the asymptotic behavior of the system. If $\lambda < 0$, the disease is eradicated at a disease-free equilibrium. In this case, we derive that the density of disease converges to 0 with exponential rate. Meanwhile, in the case $\lambda > 0$, we show that the disease is strongly stochastically permanent. It should be emphasized that the method in most papers handling periodic stochastic differential equations in the literature is to propose a Lyapunov function; and then, to use some rough estimates and stochastic inequalities to impose conditions based on that function. The conditions are therefore restrictive because of the construction of Lyapunov functions are subjective and thus, there is sizable gap between the condition for extinction and persistence (see e.g., [15,44,45]). In addition, such method is often only valid for specific models. In this paper, with the idea of analyzing the periodic measure on the boundary to determine a threshold from a dynamical point of view and proposing new techniques, we are able to give a sharp condition for extinction and persistence and only the critical case $\lambda = 0$ left untreated. The method and idea is systematic and interpretable; and thus, they can be used for other models in literature with slight modification.

The rest of the paper is arranged as follows. In Sect. 2, we give and prove our main results. Section 3.1 is reserved for providing some numerical examples and figures are provided to illustrate our results. Finally, Sect. 3.2 is devoted to discussion and further remarks for future study.

2 Main results

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space with the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual condition, i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets. We consider model (1.2) where $W(t)$ is an $\mathcal{F}_t$ -adapted Brownian motion; c, d, σ are continuous and periodic functions with period T ; $\omega, \mu, \gamma, \gamma_0$ are positive constants; $\beta(x) : [0, \infty) \mapsto \mathbb{R}_+$ is a continuous function with linear growth rate, that is, $\limsup_{x \rightarrow \infty} \frac{\beta(x)}{x} < \infty$. Let $\mathbb{R}_+^2 := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$. We first consider the equation of AQI:

$$dX(t) = (c(t) - d(t)X(t))dt + \sigma(t)\sqrt{X(t)}dW(t). \quad (2.1)$$

Theorem 2.1 *Assume that $c(t), d(t), \sigma(t)$ are positive, continuous and periodic with period T and $c(t) - \frac{\sigma^2(t)}{2} > 0$. For any initial value $x > 0$, there exists a unique globally solution. For any $k > 0$, there exist $h_1, h_2 > 0$ such that*

$$\mathbb{E}_x \left[X^k(t) - \ln X(t) + 1 \right] \leq \frac{h_2}{h_1} + (x^k - \ln x + 1)e^{-h_1 t}. \quad (2.2)$$

where the subscript in $\mathbb{E}_x$ and $\mathbb{P}_x$ indicates the initial value of (2.1). Moreover, there exist $\theta_0 > 0$, $h_3, h_4 > 0$ such that

$$\mathbb{E}_x e^{\theta_0 X(t)} \leq \frac{h_4}{h_3} + e^{-h_4 t} e^{\theta_0 x}, \quad x > 0. \quad (2.3)$$

Proof Let $V(x) = x^k - \ln x + 1$. Let $\mathcal{L}^X$ be the generator associated with the diffusion process (2.1). We have

$$\begin{aligned} [\mathcal{L}^X V](x, t) &= kx^{k-1}(c(t) - d(t)x) + k(k-1)\frac{\sigma^2(t)}{2}x^{k-1} - \frac{1}{x}(c(t) - d(t)x) + \frac{\sigma^2(t)}{2x} \\ &= -\left(c(t) - \frac{\sigma^2(t)}{2}\right)x^{-1} - kd(t)x^k + \left(k(k-1)\frac{\sigma^2(t)}{2} + k\right) + d(t) \\ &\leq -h_1 V(x) + h_2. \end{aligned} \quad (2.4)$$

Let $\tau_n := \inf\{t \geq 0 : V(X(t)) \geq n\}$. Then by Itô's formula,

$$\begin{aligned} \mathbb{E}_x e^{h_1 t \wedge \tau_n} V(X(t \wedge \tau_n)) &\leq V(x) + \mathbb{E}_x \int_0^{t \wedge \tau_n} \left(e^{h_1 s} [\mathcal{L}^X V](X(s), s) + h_1 e^{h_1 s} V(X(s)) \right) ds \\ &\leq V(x) + h_2 \mathbb{E}_x \int_0^{t \wedge \tau_n} e^{h_1 s} ds \\ &\leq V(x) + \frac{h_2}{h_1} e^{h_1 t}. \end{aligned} \quad (2.5)$$

Letting $n \rightarrow \infty$, we have

$$\mathbb{E}_x e^{h_1 t} V(X(t)) \leq V(x) + \frac{h_2}{h_1} e^{h_1 t},$$

which implies (2.2).

For $\tilde{V} = e^{\theta_0 x}$,

$$[\mathcal{L}^X \tilde{V}](x, t) = \theta \left[e^{\theta_0 x} (c(t) - d(t)x) + \frac{\theta_0 \sigma^2(t)}{2} e^{\theta_0 x} x \right] \leq h_3 - h_4 e^{\theta_0 x},$$

if $0 < \theta_0 < \min \left\{ \frac{d(t)}{\sigma^2(t)}, t \geq 0 \right\}$. Such θ_0 does always exist since $d(t)$, $\sigma(t)$ are positive, continuous and periodic functions.

Then, an application of Itô's formula, which is similar to the estimate above, implies

$$\mathbb{E}_x e^{\theta_0 X(t)} \leq \frac{h_4}{h_3} + e^{\theta_0 x} e^{h_4 t},$$

which easily completes the proof. $\square$

Since the coefficients of (2.1) are locally Lipschitz in x , it is standard to show that the solution to (2.1) is a Feller–Markov process due to the boundedness (2.2) (see [22]). An application of [18, Theorem 3.8], there exists a periodic solution $X^*(t)$ to (2.1). Moreover, because the coefficients are periodic, it is obvious that $\{X(kT) : k \in \mathbb{Z}_+\}$ is a homogeneous Markov chain. Since the diffusion is nondegenerate, for any measurable set $A \in (0, \infty)$ with positive Lebesgue measure, we have $\mathbb{P}_x\{X(T) \in A\} > 0$ for any $x \in (0, \infty)$. That together with (2.2) implies that $\{X(kT) : k \in \mathbb{Z}_+\}$ has a unique invariant probability measure π^* on $(0, \infty)$. Because of the uniqueness of the invariant probability measure of $\{X(kT) : k \in \mathbb{Z}_+\}$, the periodic $X^*(t)$ to (2.1) is unique in distribution sense and $X^*(kT)$ is π^* -distributed.

Lemma 2.1 *Let $f(x)$ be a continuous function on $[0, \infty)$ satisfying*

$$f^2(x) \leq C_f(1 + e^{\theta_2 x}),$$

for some constants $C_f > 0, 0 < \theta_2 < \theta_0$, we have

$$\mathbb{P}_x \left\{ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(X(u)) du = \mathbb{E} \frac{1}{T} \int_0^T f(X^*(u)) du = \mathbb{E}_{\pi^*} \frac{1}{T} \int_0^T f(X(u)) du \right\} = 1. \quad (2.6)$$

Proof Let $P_f(x) = \mathbb{E}_x \int_0^T f(X(s)) ds$. By the Markov property of $X(t)$, we have

$$\mathbb{E} \left[\int_{kT}^{kT+T} f(X(u)) du \middle| \mathcal{F}_{kT} \right] = P_f(X(kT)),$$

and

$$\sup_{k \in \mathbb{Z}_+} \left\{ \mathbb{E}_x \left(\int_{kT}^{kT+T} f(X(u)) du \right)^2 \right\} < \sup_{k \in \mathbb{Z}_+} \left\{ C_f T + C_f T \mathbb{E}_x \int_{kT}^{kT+T} e^{\theta_0 X(u)} du \right\} < \infty.$$

As a result, $U_k := \int_{kT}^{kT+T} f(X(u)) du - P_f(X(kT))$ is a martingale difference with respect to the filtration $\{\mathcal{F}_{kT}, k \in \mathbb{Z}_+\}$. By the strong law of large numbers for martingales, we have

$$\lim_{m \rightarrow \infty} \frac{\sum_{k=0}^m U_k}{m+1} = \lim_{m \rightarrow \infty} \left[\frac{1}{m+1} \int_0^{(m+1)T} f(X(u)) du - \frac{\sum_{k=0}^m P_f(X(kT))}{m+1} \right] = 0 \text{ a.s.}$$

for any initial value $x \in (0, \infty)$. In view of (2.3), we can deduce P_f is π^* -integrable using the arguments in [11, Lemma 3.3]. On the other hand, due to the nondegeneracy of diffusion, an application of the Feynman Kac formula and the maximum principle implies that

$$\inf_{x \in [H^{-1}, H]} \mathbb{P}_x\{X(T) \in A\} > 0,$$

for any $H > 1$ and a measurable set $A \in (0, \infty)$ which has positive Lebesgue measure (see e.g. [11, Lemma 3.6]). As a result, $\{X(kT) : k \in \mathbb{Z}_+\}$ is a Harris-recurrent Markov chain. Thus, we have from the strong law of large numbers (see e.g. [23, Theorem 7.1]) we have

$$\mathbb{P}_x \left\{ \lim_{m \rightarrow \infty} \frac{\sum_{k=0}^m P_f(X(kT))}{m+1} = \int_{(0,\infty)} P_f(x') \pi^*(dx') \right\} = 1, \text{ for all } x > 0.$$

As a result,

$$\mathbb{P}_x \left\{ \lim_{m \rightarrow \infty} \frac{1}{m+1} \int_0^{(m+1)T} f(X(u)) du = \int_{(0,\infty)} P_f(x') \pi^*(dx') \right\} = 1,$$

which clearly implies

$$\mathbb{P}_x \left\{ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(X(u)) du = \int_{(0,\infty)} P_f(x') \pi^*(dx') \right\} = 1.$$

Which together with the fact that $\mathbb{E} \frac{1}{T} \int_0^T f(X^*(u)) du = \mathbb{E}_{\pi^*} \frac{1}{T} \int_0^T f(X(u)) du$ completes the proof. $\square$

Theorem 2.2 *For any given initial value $(X(0), S(0), I(0)) \in (0, \infty) \times \mathbb{R}_+^2$, there exists a unique global solution $\{(X(t), S(t), I(t)), t \geq 0\}$ to (1.2) and the solution will remain in $(0, \infty) \times \mathbb{R}_+^2$ with probability one. The solution $\{(X(t), S(t), I(t)), t \geq 0\}$ is a Markov–Feller process on the invariant set $(0, \infty) \times \Delta$ where $\Delta = \{(s, i) \in \mathbb{R}_+^2, s + i \leq \frac{\omega}{\mu}\}$. Moreover, if $I(0) > 0$ then $I(t) > 0$ for any $t \geq 0$ with probability one.*

Proof The existence and uniqueness of $X(t)$ has been proved in Theorem 2.1. Since the coefficients in the equations of $X(t), S(t), I(t)$ are locally Lipschitz in (X, S, I) , there exists a unique locally solutions to (1.2). Moreover, the derivatives of $S(t), I(t)$ are nonnegative when $S(t), I(t)$ are 0 respectively. On the other hand, by adding side by side in system (1.2), we have

$$\begin{aligned} \frac{d}{dt}(S(t) + I(t)) &= \omega - \mu(S(t) + I(t)) - \gamma_0 I(t) \\ &\leq \omega - \mu(S(t) + I(t)). \end{aligned}$$

As a result, the solution to (1.2) with positive initial conditions cannot explode at a finite time it remains in $\mathbb{R}_+^2$. In view of the comparison theorem, if $S(0) + I(0) \leq \frac{\omega}{\mu}$, so is $(S(t) + I(t))$ for $t \geq 0$. Thus, Δ is an attracting invariant set for $(S(t), I(t))$ because if $S(0) + I(0) > K$, then $\limsup_{t \rightarrow \infty} (S(t) + I(t)) \leq K$. $\square$

In the sequel, we only work with the process $(S(t), I(t))$ on the invariant set Δ . We also denote by Δ_0 the set $\{(s, i) \in \Delta : i = 0\}$ and $\Delta^\circ = \Delta \setminus \Delta_0$. To simplify

notations, we denote by $Z(t) = (X(t), S(t), I(t))$ the solution of system (1.2), and $z = (x, s, i) \in (0, \infty) \times \Delta$. Define

$$\lambda = \frac{\omega}{\mu} \mathbb{E} \frac{1}{T} \int_0^T \beta(X^*(u)) du - \gamma - \gamma_0.$$

If $\lambda > 0$, we define

$$U(z) = e^{\theta_1 x} \ln \frac{x+1}{x} i^{-\ell \theta_1},$$

for some small θ_1, ℓ as in the following Lemma. The idea of choosing this function U as follows. The term $i^{-\ell \theta_1}$ will manage the behavior of $I(t)$ near the boundary. It is the main term because we focus on the dynamics of $I(t)$ when it is small. However, since the dynamics of $I(t)$ depends on $X(t)$, we need some terms to control the dynamics of $X(t)$ when it is too small or too large. The properties in (2.2) and (2.3), suggests us to use $e^{\theta_1 x}$ and $\ln x$ to control the dynamics of $X(t)$ near ∞ and near 0 respectively. The term $\ln(1+x)$ is used to diminish the effect of $\ln x$ when x is large so that we can focus on $e^{\theta_1 x}$ as x large. The constant θ_1, ℓ are chosen appropriately such that the function U satisfies the desired property (2.7) and (2.8) below.

Lemma 2.2 *For sufficiently small $\theta_1 > 0$ and $\ell > 0$, there exist $h_6 > 0$ and $M > 0$ such that*

$$\mathbb{E}_z U(Z(t_2)) \leq e^{h_6(t_2-t_1)} \mathbb{E}_z U(Z(t_1)), \quad \text{for } t_1 < t_2, \quad (2.7)$$

and

$$\mathbb{E}_z U(Z(T)) \leq \rho U(z), \quad \text{if } x \vee x^{-1} \geq M, \quad (2.8)$$

where $\rho \in (0, 1)$ is some constant, and the subscript of $\mathbb{P}_z$ and $\mathbb{E}_z$ indicates the initial value of $Z(t)$.

Proof Let $\mathcal{L}$ be the generator associated with the diffusion (1.2). Directed calculation shows that

$$\begin{aligned} [\mathcal{L}U](z, t) = & i^{-\ell \theta_1} e^{\theta_1 x} \ln \frac{x+1}{x} \left(\theta_1 c(t) - \theta_1 d(t)x + \frac{\theta_1^2 \sigma^2(t)}{2} x - \ell \theta_1 (s\beta(x) - \mu - \gamma_0 - \gamma) \right) \\ & + i^{-\ell \theta_1} e^{\theta_1 x} \left(-\frac{(c(t) - d(t)x)}{x(x+1)} + \left(\frac{1}{x} - \frac{x}{(x+1)^2} \right) \frac{\sigma^2(t)}{2} \right) - \theta_1 i^{-\ell \theta_1} e^{\theta_1 x} \frac{\sigma^2}{x+1}. \end{aligned}$$

Let $x^* := \max_{t \geq 0} \frac{2[c(t) + (\mu + \gamma_0 + \gamma)]}{d(t)}$. There are small $\theta_1, \ell > 0$ satisfying

$$\ell, \theta_1 < 1, \quad \theta_1 < \min_{t \geq 0} \frac{d(t)}{\sigma^2(t)},$$

such that we have

$$h_5 := \frac{1}{2} \theta_1 \inf_{x > x^*, t \geq 0} \left\{ d(t)x - \theta_1 \frac{\sigma^2(t)}{2} x - c(t) - \ell \theta_1 (\mu + \gamma_0 + \gamma) \right\} > 0,$$

and

$$m_1 := \sup_{x \geq 0, t \geq 0} \left(\theta_1 c(t) - \theta_1 d(t)x + \frac{\theta_1^2 \sigma^2(t)}{2} x - \ell \theta_1 (s\beta(x) - \mu - \gamma_0 - \gamma) \right) < \infty.$$

Moreover, because $\min_{t \geq 0} \left\{ c(t) - \frac{\sigma^2(t)}{2} \right\} > 0$ and $\lim_{x \rightarrow 0} \frac{\ln \frac{x+1}{x}}{\frac{x}{x+1}} = 0$, we have

$$-\frac{(c(t) - d(t)x)}{x(x+1)} + \left(\frac{1}{x} - \frac{x}{(x+1)^2} \right) \frac{\sigma^2(t)}{2} \leq -(h_5 + m_1) \ln \frac{x+1}{x}, \quad (2.9)$$

when x is sufficiently small. On the other hand,

It follows from the choice of θ_1 , ℓ and definition of h_5 , m_1 and (2.9) that

$$[\mathcal{L}U](z, t) \leq -h_5 U(z), \text{ if } x \vee x^{-1} \geq K_1 \text{ for some } K_1 > 0.$$

Moreover, we have $\sup_{\{x \vee x^{-1} \leq K_1\}} \frac{[\mathcal{L}(U)](z, t) + h_5 U(z)}{U(z)} =: h_6 < \infty$. As a result,

$$[\mathcal{L}U](z, t) \leq \mathbf{1}_{\{x \vee x^{-1} < K_1\}} h_6 U(z) - h_5 U(z), z \in (0, \infty) \times \Delta^\circ, \quad (2.10)$$

where $\mathbf{1}_A$ is the indicator function of the set A . Let $\eta := \inf\{t > 0 : X(t) > K_1 \text{ or } X(t) < K_1^{-1}\}$. We have from Itô's formula and (2.10) that

$$\mathbb{E}_z e^{h_5(T \wedge \eta)} U(Z(T \wedge \eta)) \leq U(z), \quad (2.11)$$

and

$$\mathbb{E}_z U(Z(t_2)) \leq e^{h_6(t_2 - t_1)} \mathbb{E}_z U(Z(t_1)), \text{ for } t_1 < t_2, \quad (2.12)$$

which together with the Markov–Feller property of $X(t)$ implies

$$\mathbb{E}_z \mathbf{1}_{\{\eta < T\}} e^{-h_6(T - \eta)} U(Z(T)) \leq \mathbb{E}_z \mathbf{1}_{\{\eta < T\}} U(Z(\eta)) \leq \max_{x \vee x^{-1} \leq K_1} e^{\theta_1 x} \ln \frac{x+1}{x} =: u_1. \quad (2.13)$$

From (2.11), we have

$$\mathbb{E}_z \mathbf{1}_{\{\eta \geq T\}} e^{h_5 T} U(Z(T)) \leq U(z)$$

or, equivalently

$$\mathbb{E}_z \mathbf{1}_{\{\eta \geq T\}} U(Z(T)) \leq e^{-h_5 T} (U(z)). \quad (2.14)$$

Which together with (2.13) imply that

$$\mathbb{E}_z U(Z(T)) \leq e^{-h_5 T} U(z) + u_1 e^{h_6 T}. \quad (2.15)$$

Since $\lim_{x \vee x^{-1} \rightarrow \infty} U(z) = \infty$, we can easily deduce (2.8) from (2.15). $\square$

Lemma 2.3 Let $f(z)$ be a continuous function on $[0, \infty) \times \Delta$, satisfying

$$|f(z)| \leq K_f(1 + e^{\theta_2 x})$$

for some constants $K_f > 0$, $0 < \theta_2 < \theta_0$. Then, for any $H > 1$, we have the uniform convergence:

$$\lim_{t \rightarrow \infty} \sup_{\{z \in [H^{-1}, H] \times \Delta_0\}} \left\{ \mathbb{E}_z \frac{1}{t} \int_0^t f(Z(u)) du - \mathbb{E}_{\pi^*} \frac{1}{T} \int_0^T f \left(X^*(u), \frac{\omega}{\mu}, 0 \right) du \right\} = 0 \quad (2.16)$$

Proof Consider the occupation measure,

$$\Pi_n^z(\cdot) = \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{\{Z(kT) \in \cdot\}}.$$

Due to Lemma 2.2 and the fact that $(S(t), I(t))$ lives in a compact space, the family $\{\Pi_n^z, z \in [H^{-1}, H] \times \Delta_0, n \in \mathbb{Z}_+\}$ is tight. Moreover, it is well-known (see e.g. [29, Lemma 6]) that any weak-limit of Π_n^z as $n \rightarrow \infty$ is an invariant measure of $\{Z(kT), k \in \mathbb{Z}_+\}$. Moreover, with any initial value in $(0, \infty) \times \Delta_0$, we have $\lim_{t \rightarrow \infty} (S(t), I(t)) = \left(\frac{\omega}{\mu}, 0\right)$ and π^* is the unique invariant measure of $\{X(kT), k \in \mathbb{Z}_+\}$. Thus, $\pi^* \times \delta_{(\frac{\omega}{\mu}, 0)}$ is the unique invariant probability measure of $\{Z(kT), k \in \mathbb{Z}_+\}$ on $(0, \infty) \times \Delta_0$, where $\delta_{(\frac{\omega}{\mu}, 0)}$ is the Dirac measure with mass at $\left(\frac{\omega}{\mu}, 0\right)$. As a result, for any bounded and continuous function on $(0, \infty) \times \Delta$, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E}_z \frac{1}{t} \int_0^t g(Z(u)) du &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum \mathbb{E}_{Z(kT)} \frac{1}{T} \int_0^T g(Z(u)) du \\ &= \mathbb{E}_{\pi^*} \frac{1}{T} \int_0^T g \left(X(u), \frac{\omega}{\mu}, 0 \right) du. \end{aligned}$$

Now, assume that $f(z)$ is continuous and $|f(z)| \leq K_f(1 + e^{\theta_2 x})$

For any $H > 0$, consider the truncated function: $f_H(z) = -H \vee (f(z) \wedge H)$. We have

$$|f(z) - f_H(z)| \leq |f(z)| \mathbf{1}_{\{|f(z)| \geq H\}} \leq \frac{|f(z)|^{1+\gamma_f}}{H^{\check{\gamma}}} \leq \frac{(K_f(1 + e^{\theta_2 x})^{1+\gamma_f})}{H^{\check{\gamma}}} \leq \frac{\check{K}_f(1 + e^{\theta_0 x})}{H^{\check{\gamma}}},$$

for $0 < \check{\gamma} \leq \frac{\theta_0}{\theta_1}$ and $\check{K}_f$ is a constant depending on K_f and $\check{\gamma}$, which together with (2.3) implies

$$\lim_{t \rightarrow \infty} \sup \mathbb{E}_z \frac{1}{t} \int_0^t |f(Z(u)) - f_H(Z(u))| du$$

$$\leq \limsup_{t \rightarrow \infty} \frac{\check{K}_f}{H^{\check{\gamma}}} \mathbb{E}_z \frac{1}{t} \int_0^t (1 + e^{\theta_0 X(u)}) du \leq \frac{\check{K}_f}{H^{\check{\gamma}}} \left(1 + \frac{h_4}{h_3}\right).$$

On the other hand, since $f_H(z)$ is continuous and bounded, we have

$$\lim_{t \rightarrow \infty} \mathbb{E}_z \frac{1}{t} \int_0^t f_H(Z(u)) du = \mathbb{E}_{\pi^*} \frac{1}{T} \int_0^T f_H\left(X(u), \frac{\omega}{\mu}, 0\right) du.$$

Combining two limits above and let H go to infinity, we easily obtain the uniform convergence (2.16). $\square$

Lemma 2.4 *Let M, ρ be as in Lemma 2.2 and $n^* \in \mathbb{Z}_+$ such that $e^{h_6 T} \rho^{n^* - 1} < 1$. There exist $\tilde{\lambda} > 0, \theta \in (0, 1), K_\theta > 0, m^* \in \mathbb{Z}_+$ such that and*

$$\mathbb{E}_z U^\theta(kT^*) \leq \exp\{-\tilde{\lambda}\theta kT^*\} U^\theta(z)z + K_\theta, \quad \forall z \in [M^{-1}, M] \times \Delta^\circ, k \in \{0, 1, \dots, n^*\} \quad (2.17)$$

where $T^* = m^*T$.

Proof On the other hand, since $|\ln(\ln \frac{x+1}{x})| \leq C(x+1 - \ln x)$, we have from (2.2) that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \left(\mathbb{E}_x \ln \left(\ln \frac{X(t)+1}{X(t)} \right) - \ln \left(\ln \frac{x+1}{x} \right) \right) = 0, \quad (2.18)$$

and

$$\lim_{t \rightarrow \infty} \frac{1}{t} (\mathbb{E}_x X(t) - x) = 0, \quad (2.19)$$

uniformly in each compact set (of $(0, \infty)$) of initial values.

In view of Lemma 2.3, we can find $m^* > 0$ sufficiently large such that $e^{h_6 T} \rho^{n^*} < 1$ [because $\rho \in (0, 1)$] and

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{E}_z \frac{1}{\tilde{T}} \int_0^{\tilde{T}} (\beta(X(t))S(t) - \mu - \gamma - \gamma_0) dt \\ & \geq \frac{3\lambda}{4} \text{ for any } z \in [M^{-1}, M] \times \Delta_0, \tilde{T} \geq T^* := m^*T. \end{aligned}$$

Due to (2.18) and (2.19), we can assume that m^* be sufficiently large that

$$\frac{\theta_1 \ell}{\tilde{T}} \left(\mathbb{E}_x \ln \left(\ln \frac{X(\tilde{T})+1}{X(\tilde{T})} \right) - \ln \left(\ln \frac{x+1}{x} \right) \right) + \frac{\theta_1}{\tilde{T}} (\mathbb{E}_x X(\tilde{T}) - x) \leq \frac{\theta_1 \ell \lambda}{4}, \quad (2.20)$$

for $x \in [M^{-1}, M]$ and $\tilde{T} \geq T^*$. In view of the Markov–Feller property of $Z(t)$, we have

$$\mathbb{E}_z \frac{1}{\tilde{T}} (\ln I(\tilde{T}) - i) = \mathbb{E} \frac{1}{\tilde{T}} \int_0^{\tilde{T}} (\beta(X(t))S(t) - \mu - \gamma - \gamma_0) dt \geq \frac{\lambda}{2}, \quad (2.21)$$

for any $z \in [M^{-1}, M] \times \Delta_\delta^\circ$, $\tilde{T} \in [T^*, n^*T^*]$ and for some $\delta > 0$, where $\Delta_\delta^\circ = \{(s, i, r) \in \Delta, 0 < i < \delta\}$. Combining (2.20) and (2.21) yields

$$\mathbb{E}_z[\ln U(Z(n^*T^*)) - \ln U(z)] \leq \frac{\lambda \ell \theta_1}{4} n^*T^*, z \in [M^{-1}, M] \times \Delta_\delta^\circ.$$

Applying [11, Lemma 3.5], we have

$$\begin{aligned} \mathbb{E}_z U^\theta(\tilde{T}) &\leq \exp\left\{-\frac{\lambda \ell \theta_1}{8} \theta \tilde{T}\right\} U^\theta(z) := \exp\{-\tilde{\lambda} \theta \tilde{T}\} U^\theta(z), \\ \text{for } z &\in [M^{-1}, M] \times \Delta_\delta^\circ, \tilde{T} \in [T^*, n^*T^*] \end{aligned}$$

for sufficiently small $\theta > 0$. This and (2.7) imply (2.17) with

$$K_\theta = e^{h_6 n^* T^*} \sup_{\{M^{-1} \leq x \leq M, i \geq \delta, (s, i) \in \Delta\}} U(z) < \infty.$$

□

Theorem 2.3 Suppose $\lambda < 0$. Let n^*, θ be as in Lemma 2.4. There exists $C > 0$ such that

$$\limsup_{t \rightarrow \infty} \mathbb{E}_z U^\theta(Z(t)) \leq C, \text{ for any } z \in (0, \infty) \times \Delta^\circ. \quad (2.22)$$

As a result, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\liminf_{t \rightarrow \infty} \mathbb{P}_z\{I(t) \geq \delta\} \geq 1 - \varepsilon$$

for any $z \in (0, \infty) \times \Delta^\circ$.

Proof The proof is given by developing the techniques in [11], which is used to treat homogeneous Markov processes. Define

$$\tau = \inf\{t = kT \geq 0 : X(t) \vee X^{-1}(t) \leq M, k \in \mathbb{Z}_+\}. \quad (2.23)$$

Due to (2.8) and the Markov property of $\{Z(nT)\}$, the process $\{\rho_2^{-(kT) \wedge \tau} U(Z(kT)), k \in \mathbb{Z}_+\}$ is a supermartingale with respect to the filtration $\{\mathcal{F}_{kT}, k \in \mathbb{Z}_+\}$. Hence

$$\mathbb{E}_z \left[\rho^{-\theta(\tau \wedge n^*T^*)} U^\theta(Z(\tau \wedge n^*T^*)) \right] \leq U^\theta(z), z \in (0, \infty) \times \Delta^\circ.$$

Thus, one has

$$\begin{aligned}
 U^\theta(z) &\geq \mathbb{E}_z \left[\rho^{-\theta(\tau \wedge n^* T^*)} U^\theta(Z(\tau \wedge n^* T^*)) \right] \\
 &= \mathbb{E}_z \left[\mathbf{1}_{\{\tau \leq (n^*-1)T^*\}} \rho^{-\theta(\tau \wedge n^* T^*)} U^\theta(Z(\tau \wedge n^* T^*)) \right] \\
 &\quad + \mathbb{E}_z \left[\mathbf{1}_{\{(n^*-1)T^* < \tau < n^* T^*\}} \rho^{-\theta(\tau \wedge n^* T^*)} U^\theta(Z(\tau \wedge n^* T^*)) \right] \\
 &\quad + \mathbb{E}_z \left[\mathbf{1}_{\{\tau \geq n^* T^*\}} \rho^{-\theta(\tau \wedge n^* T^*)} U^\theta(Z(\tau \wedge n^* T^*)) \right] \quad (2.24) \\
 &\geq \mathbb{E}_z \left[\mathbf{1}_{\{\tau \leq (n^*-1)T^*\}} U^\theta(Z(\tau)) \right] \\
 &\quad + \rho^{-\theta(n^*-1)T^*} \mathbb{E}_z \left[\mathbf{1}_{\{(n^*-1)T^* < \tau < n^* T^*\}} U^\theta(Z(\tau)) \right] \\
 &\quad + \rho^{-\theta n^* T^*} \mathbb{E}_z \left[\mathbf{1}_{\{\tau \geq n^* T^*\}} U^\theta(Z(n^* T^*)) \right].
 \end{aligned}$$

By the strong Markov property of $Z(t)$ and Lemma 2.4, we obtain

$$\begin{aligned}
 &\mathbb{E}_z \left[\mathbf{1}_{\{\tau \leq (n^*-1)T^*\}} U^\theta(Z(n^* T^*)) \right] \\
 &\leq \mathbb{E}_z \left[\mathbf{1}_{\{\tau \leq (n^*-1)T^*\}} \left[K_\theta + e^{-\tilde{\lambda}\theta(n^* T^* - \tau)} U^\theta(Z(\tau)) \right] \right] \quad (2.25) \\
 &\leq K_\theta + \exp(-\tilde{\lambda}\theta T^*) \mathbb{E}_z \left[\mathbf{1}_{\{\tau \leq (n^*-1)T^*\}} U^\theta(Z(\tau)) \right]
 \end{aligned}$$

Similarly, we deduce from the strong Markov property of $Z(t)$, Jensen's inequality and Lemma 2.2 that

$$\begin{aligned}
 &\mathbb{E}_z \left[\mathbf{1}_{\{(n^*-1)T^* < \tau < n^* T^*\}} U^\theta(Z(n^* T^*)) \right] \\
 &\leq \mathbb{E}_z \left[\mathbf{1}_{\{(n^*-1)T^* < \tau < n^* T^*\}} h_6^{\theta(n^* T^* - \tau)} U^\theta(Z(\tau)) \right] \quad (2.26) \\
 &\leq h_6^{\theta T^*} \mathbb{E}_z \left[\mathbf{1}_{\{(n^*-1)T^* < \tau < n^* T^*\}} U^\theta(Z(\tau)) \right].
 \end{aligned}$$

Applying (2.25) and (2.26) to (2.24) yields

$$\begin{aligned}
 U^\theta(x) &\geq \mathbb{E}_z \left[\mathbf{1}_{\{\tau \leq (n^*-1)T^*\}} U^\theta(Z(\tau)) \right] \\
 &\quad + \rho^{-\theta(n^*-1)T^*} \mathbb{E}_z \left[\mathbf{1}_{\{(n^*-1)T^* < \tau < n^* T^*\}} U^\theta(Z(\tau)) \right] \\
 &\quad + \rho^{-\theta n^* T^*} \mathbb{E}_z \left[\mathbf{1}_{\{\tau \geq n^* T^*\}} U^\theta(Z(n^* T^*)) \right] \text{ (due to (2.24))} \\
 &\geq \exp(\tilde{\lambda}\theta T^*) \mathbb{E}_z \left[\mathbf{1}_{\{\tau \leq (n^*-1)T^*\}} U^\theta(Z(n^* T^*)) \right] - \exp(\tilde{\theta} T^*) K_\theta \quad (2.27) \\
 &\quad + h_6^{-\theta T^*} \rho^{-\theta(n^*-1)T^*} \mathbb{E}_z \left[\mathbf{1}_{\{(n^*-1)T^* < \tau < n^* T^*\}} U^\theta(Z(n^* T^*)) \right] \\
 &\quad + \rho^{-\theta n^* T^*} \mathbb{E}_z \left[\mathbf{1}_{\{\tau \geq n^* T^*\}} U^\theta(Z(n^* T^*)) \right] \text{ (due to (2.25) and (2.26))} \\
 &\geq \kappa^{-1} \mathbb{E}_z U^\theta(Z(n^* T^*)) - K_\theta \exp\left(\frac{1}{2}\theta \rho^* T^*\right)
 \end{aligned}$$

where $\kappa = \max \left\{ \exp(-\tilde{\lambda}\theta T^*), h_6^{-\theta T} \rho^{\theta(n^*-1)}, \rho^{\theta n^*} \right\} < 1$ (due to the definition of n^*). The proof of (2.22) is completed by taking

$$\tilde{K} = K_\theta \exp\left(\frac{1}{2}\theta\rho^*T\right)\kappa.$$

Then, we have $\mathbb{E}_z U^\theta(Z(n^*T^*)) \leq \kappa U^\theta(z) + \tilde{K}$ which leads to

$$\mathbb{E}_z U^\theta(Z(kn^*T^*)) \leq \kappa^k U^\theta(z) + \frac{\tilde{K}}{1-\kappa}. \quad (2.28)$$

In view of (2.7) and the Markov, we have

$$\sup_{t \in [kn^*T^*, (k+1)n^*T^*]} \mathbb{E}_z U^\theta(Z(t)) \leq e^{h_6 n^* T^*} \mathbb{E}_z U^\theta(Z(n^*T^*)). \quad (2.29)$$

By virtue of (2.28) and (2.29), we have

$$\limsup_{t \rightarrow \infty} \mathbb{E}_z U^\theta(Z(t)) \leq e^{h_6 n^* T^*} \frac{\tilde{K}}{1-\kappa},$$

which is the first claim of the theorem. The second claim easily follows from an application of Markov's inequality to (2.22). $\square$

Theorem 2.4 *If $\lambda < 0$, then $\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq \lambda < 0$ almost surely for any initial value.*

Proof We have that

$$\begin{aligned} \limsup \frac{\ln I(t)}{t} &= \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t (\beta(X(u))S(u) - \mu - \gamma - \gamma_0) du \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left(\beta(X(u)) \frac{\omega}{\mu} - \mu - \gamma - \gamma_0 \right) du \\ &= \lambda < 0 \text{ a.s. (due to Lemma 2.1).} \end{aligned}$$

The proof is complete. $\square$

3 Examples and discussion

3.1 Examples

Next, we illustrate our findings by some numerical examples. In the first example, $c(t) = 2 + \sin(\pi t)$, $d(t) = 2 + \cos(\pi t)$, $\sigma(t) = 1$, $\omega = 2$, $\mu = 1$, $\gamma = 0.09$, $\gamma_0 = 0.01$ and $\beta(x) = \frac{2x}{x+1}$. We are going consider the saturated incidence rates, which are increasing and but bounded. That is why we consider the above $\beta(x)$.

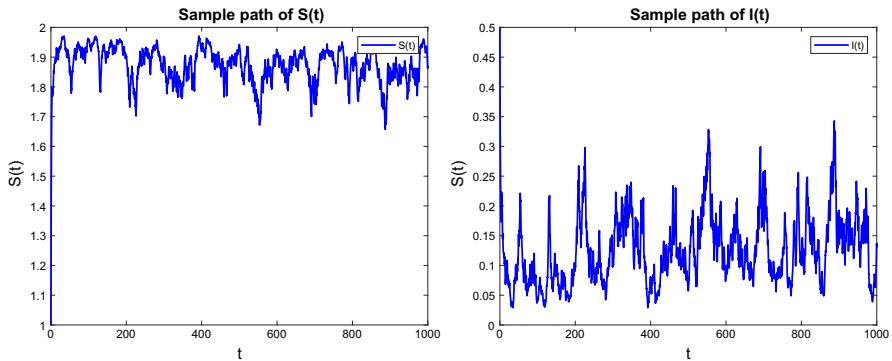


Fig. 1 Sample paths of $S(t)$ and $I(t)$

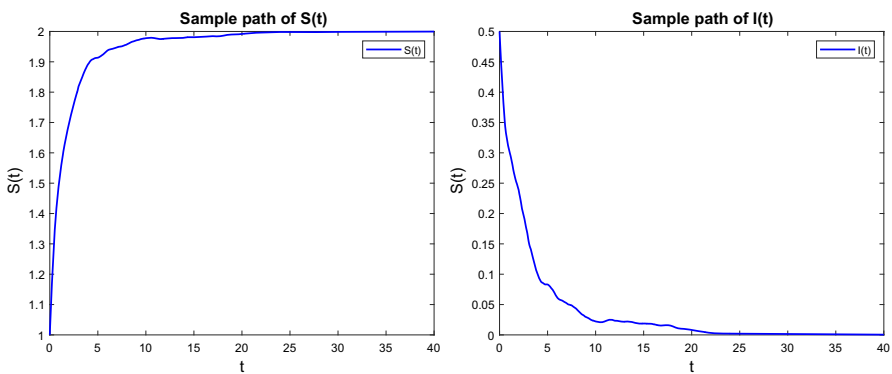


Fig. 2 Sample paths of $S(t)$ and $I(t)$

The threshold λ is approximated by $\frac{1}{t} \int_0^t \beta(X(u)) du$ with $t = 10,000$. The approximate value of λ is 0.08. The disease persists as illustrated in Fig. 1.

In the second example, we keep the same set of parameters as in the first example, except for $\beta(x)$, which is now given by $\beta(x) = \frac{0.8x}{x+1}$. The approximate value of λ is $-0.63 < 0$. As a result, $I(t)$ converges to 0 exponentially fast, see Fig. 2.

3.2 Discussion

The effects of air pollution on respiratory infection were discussed in [10,34] with focus on parameter estimation, simulation and prediction. Motivated by these works, we focus on examining mathematically long-term properties of a polluted SIS model with periodic parameters. The theoretical results in this paper will facilitate prediction and estimation. We classify the longtime behavior (persistence and extinction of disease) by introducing a threshold λ obtained from dynamical point of view.

Moreover, the mathematical techniques in this paper is novel and can be applied to many periodic stochastic epidemiological models. Note that a common approach to handle these types of models is as follows:

- Propose a Lyapunov function and impose some conditions to obtain sufficient conditions for the persistence of the disease
- Use some rough estimates and stochastic inequalities to prove the extinction of the disease under certain conditions.

Both techniques do not really reflect the dynamics of the systems and therefore, the obtained results are limited and the conditions are restrictive. In other words, there is a big gap between conditions for extinction and persistence; see [14,19–21,26,39] for example. In contrast, we analyze the dynamics of the systems on the boundary (by looking at the periodic invariant measure on the boundary) to determine the threshold value for the persistence and persistence of the disease without any additional conditions on the parameters. Our results are sharp in the sense that only the critical case $\lambda = 0$ is untreated.

Moreover, the techniques in the proof of Theorem 2.3 are also original because one has to nicely handle the behaviors when the pollution is at a small level or a large level. These techniques can be applied to many other periodic stochastic models (see e.g., [14,19–21,26,39]) and to obtain sharp results.

A promising direction for future study is to take individual awareness into consideration, see e.g., [32,33]. Under the impact of awareness, the transmission of the disease depends on the quality of the information available to a given susceptible individual. That will make the infection rate and the model more complicated. Study the longtime behavior in this case would be much interesting in both theoretical and applications point of view.

Another direction is to treat stochastic periodic systems with delays. The main difficulty is that we must work on an infinite dimensional space. The results developed in [37,40–43] will play a key role in handling stochastic delay systems in epidemiology. That would be a future research direction.

Acknowledgements The authors would like to thank the anonymous reviewers and the editor for their constructive comments and remarks, which greatly improved the quality of this paper.

References

1. Bernstein, J.A., Alexis, N., Barnes, C., Bernstein, I.L., Nel, A., Peden, D., Diaz-Sanchez, D., Tarlo, S.M., Williams, P.B.: Health effects of air pollution. *J. Allergy Clin. Immunol.* **114**, 1116–1123 (2004)
2. Bichara, D., Iggridr, A., Sallet, G.: Global analysis of multi-strains SIS, SIR and MSIR epidemic models. *J. Appl. Math. Comput.* **44**(1–2), 273–292 (2014)
3. Botchev, M.A., Verwer, J.G.: A new approximate matrix factorization for implicit time integration in air pollution modeling. *J. Comput. Appl. Math.* **157**, 309–327 (2003)
4. Chan, L.G., Parashar, U.D., Lye, M.S., Ong, F.G., Zaki, S.R., Alexander, J.P., Ho, K.K., Han, L.L., Pallansch, M.A., Suleiman, A.B., Jegathesan, M., Anderson, L.J.: Deaths of children during an outbreak of hand, foot, and mouth disease in Sarawak, Malaysia: clinical and pathological characteristics of the disease. *Clin. Infect. Dis.* **31**(3), 678–683 (2000)
5. Dieu, N.T., Du, N.H., Nhu, N.N.: Conditions for permanence and ergodicity of certain SIR epidemic models. *Acta. Appl. Math.* **160**, 81–99 (2019)
6. Dieu, N.T., Nguyen, D.H., Du, N.H., Yin, G.: Classification of asymptotic behavior in a stochastic SIR model. *SIAM J. Appl. Dyna. Syst.* **15**, 1062–1084 (2016)
7. Du, N.H., Nhu, N.N.: Permanence and extinction of certain stochastic SIR models perturbed by a complex type of noises. *Appl. Math. Lett.* **64**, 223–230 (2017)

8. Du, N.H., Nhu, N.N.: Permanence and extinction for the stochastic SIR epidemic model. [arXiv:1812.03333](https://arxiv.org/abs/1812.03333)
9. Finkbeiner, S.R., Allred, A.F., Tarr, P.I., Klein, E.J., Kirkwood, C.D., Wang, D.: Metagenomic analysis of human diarrhea: viral detection and discovery. *PLoS Pathog.* **4**(2), 1–9 (2008)
10. He, S., Tang, S., Xiao, Y.: Stochastic modelling of air pollution impacts on respiratory infection risk. *Bull. Math. Biol.* **80**, 3127–3153 (2018)
11. Hening, A., Dang, N.H.: Coexistence and extinction for stochastic Kolmogorov systems. *Ann. Appl. Probab.* **28**(3), 1893–1942 (2018)
12. Hieu, N.T., Du, N.H., Auger, P., Dang, N.H.: Dynamical behavior of a stochastic sirs epidemic model. *Math. Model. Nat. Phenom.* **10**, 56–73 (2015)
13. Ji, Y.: Economic, growth, urbanization and air pollution in China: an empirical research based on panel data. *Energies* **7**(7), 4202–4220 (2014)
14. Ji, C., Jiang, D.: The threshold of a non-autonomous SIRS epidemic model with stochastic perturbations. *Math. Methods Appl. Sci.* **40**, 1773–1782 (2017)
15. Jiang, D., Zhang, Q., Hayat, T., Alsaedi, A.: Periodic solution for a stochastic non-autonomous competitive Lotka–Volterra model in a polluted environment. *Phys. A* **471**, 276–287 (2017)
16. Kermack, W.O., McKendrick, A.G.: Contributions to the mathematical theory of epidemics (part I). *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **115**, 700–721 (1927)
17. Kermack, W.O., McKendrick, A.G.: A contribution to the mathematical theory of epidemics—the problem of endemicity (part II). *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **138**, 55–83 (1932)
18. Khasminskii, R.: *Stochastic Stability of Differential Equations*. Springer, Berlin (2011)
19. Liu, Q., Jiang, D., Shi, N., Hayat, T., Tasawar, A., Alsaedi, A.: Periodic solution for a stochastic nonautonomous SIR epidemic model with logistic growth. *Phys. A* **462**, 816–826 (2016)
20. Liu, Q., Jiang, D., Shi, N., Hayat, T., Alsaedi, A.: Nontrivial periodic solution of a stochastic non-autonomous SISV epidemic model. *Phys. A* **462**, 837–845 (2016)
21. Liu, Q., Jiang, D., Hayat, T., Ahmad, B.: Periodic solution and stationary distribution of stochastic SIR epidemic models with higher order perturbation. *Phys. A* **482**, 209–217 (2017)
22. Mao, X.: *Stochastic Differential Equations and Applications*. Elsevier, Amsterdam (2007)
23. Meyn, S.P., Tweedie, R.L.: Stability of Markovian processes I: criteria for discrete-time chains. *Adv. Appl. Probab.* **24**, 542–574 (1992)
24. Nguyen, D., Nguyen, N., Yin, G.: Analysis of a spatially inhomogeneous stochastic partial differential equation epidemic model. *J. Appl. Probab.* (to appear). [arXiv:1812.03326](https://arxiv.org/abs/1812.03326)
25. Nguyen, N.N., Yin, G.: Stochastic partial differential equation SIS epidemic models: modeling and analysis. *Commun. Stoch. Anal.* **13**, 8 (2019)
26. Qi, H., Liu, L., Meng, X.: Dynamics of a nonautonomous stochastic SIS epidemic model with double epidemic hypothesis. *Complexity* **2017**, 4861391 (2017). <https://doi.org/10.1155/2017/4861391>
27. Rajasekar, S.P., Pitchaimani, M., Zhu, Q.: Dynamic threshold probe of stochastic SIR model with saturated incidence rate and saturated treatment function. *Phys. A* **535**, 122300 (2019)
28. Rajasekar, S.P., Pitchaimani, M., Zhu, Q.: Progressive dynamics of a stochastic epidemic model with logistic growth and saturated treatment. *Phys. A* **538**, 122649 (2020)
29. Schreiber, S.J., Benaïm, M., Atchadé, K.: Persistence in fluctuating environments. *J. Math. Biol.* **62**, 655–683 (2011)
30. Shang, Y.: A Lie algebra approach to susceptible-infected-susceptible epidemics. *Electron. J. Differ. Equ.* **233**, 1–7 (2012)
31. Shang, Y.: Analytical solution for an in-host viral infection model with time-inhomogeneous rates. *Acta Phys. Polon. B* **46**, 1567–1577 (2015)
32. Shang, Y.: Degree distribution dynamics for disease spreading with individual awareness. *J. Syst. Sci. Complex* **28**, 96–104 (2015). <https://doi.org/10.1007/s11424-014-2186-x>
33. Shang, Y.: Modeling epidemic spread with awareness and heterogeneous transmission rates in networks. *J. Biol. Phys.* **39**(3), 96–104 (2015)
34. Tang, S.Y., Yan, Q.L., Shi, W., Wang, X., Sun, X.D., Yu, P.B., Wu, J.H., Xiao, Y.N.: Measuring the impact of air pollution on respiratory infection risk in China. *Environ. Pollut.* **232**, 477–486 (2018)
35. Tuong, T.D., Dang, N.H., Dieu, N.T., Ky, T.Q.: Extinction and permanence in a stochastic SIRS model in regime switching with general incidence rate. *Nonlinear Anal. Hybrid Syst.* **34**(2019), 121–130 (2019)

36. Verma, R., Tiwari, S.P., Upadhyay, R.K.: Transmission dynamics of epidemic spread and outbreak of Ebola in West Africa: fuzzy modeling and simulation. *J. Appl. Math. Comput.* **60**(1–2), 637–671 (2019)
37. Wang, B., Zhu, Q.: Stability analysis of semi-Markov switched stochastic systems. *Automatica* **94**, 72–80 (2018)
38. Wei, T., Lin, P., Zhu, Q., Wang, L., Wang, Y.: Dynamical behavior of nonautonomous stochastic reaction-diffusion neural-network models. *IEEE Trans. Neural Netw. Learn. Syst.* **30**, 1575–1580 (2019)
39. Zhang, W., Meng, X.: Stochastic analysis of a novel nonautonomous periodic SIRS epidemic system with random disturbances. *Phys. A* **492**, 1290–1301 (2018)
40. Zhu, Q.: Stabilization of stochastic nonlinear delay systems with exogenous disturbances and the event-triggered feedback control. *IEEE Trans. Autom. Control* **64**(9), 3764–3771 (2019)
41. Zhu, Q.: Stability analysis of stochastic delay differential equations with Lévy noise. *Syst. Control Lett.* **118**, 62–68 (2018)
42. Zhu, Q., Wang, H.: Output feedback stabilization of stochastic feedforward systems with unknown control coefficients and unknown output function. *Automatica* **87**, 166–175 (2018)
43. Hu, W., Zhu, Q., Karimi, H.R.: Some improved Razumikhin stability criteria for impulsive stochastic delay differential systems. *IEEE Trans. Autom. Control* **64**(12), 5207–5213 (2019)
44. Zu, L., Jiang, D., O'Regan, D.: Periodic solution for a stochastic non-autonomous predator-prey model with Holling II functional response. *Acta Appl. Math.* **161**, 89–105 (2019)
45. Zuo, W., Jiang, D.: Periodic solutions for a stochastic non-autonomous Holling–Tanner predator-prey system with impulses. *Nonlinear Anal. Hybrid Syst.* **22**, 191–201 (2016)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.